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THE MATHEMATICS TEACHER

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SOME APPLICATIONS OF ALGEBRA TO THEOREMS IN SOLID GEOMETRY

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The student in the secondary school is likely to feel when studying geometry that the methods of analysis used are far removed from those used in the study of other branches of mathematics. Some recent authors have made extensive use of elementary algebra in texts on geometry. The scope of algebra employed could well be extended to the advantage of the student in both subjects. When once a student sees a useful application of a particular topic in algbra its value is immediately apparent and his interest and efforts are stimulated. It is the purpose of this article to draw the attention of teachers to certain neat applications of limiting values of fractions and summation of series to theorems in solid geometry.

Use will be made of the limiting value of the fraction $\frac{a}{n}$, in which a is any constant, as n becomes large beyond bound or

$\displaystyle\lim_{n \to \infty} \frac{a}{n} = 0$; of the sum of the arithmetic series

$$1 + 2 + 3 + \quad . \quad . \quad . \quad + n = \frac{n(n+1)}{2}$$

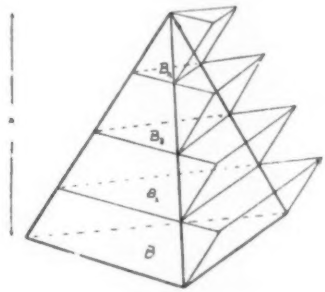
and of the sum of the squares of the natural numbers, the series

$$1^2 + 2^2 + 3^2 + \quad . \quad . \quad . \quad + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

This last summation may be readily effected by the method of undetermined coefficients or by the method of summation by differences, or by mathematical induction.

1. To find the volume V of a triangular pyramid of base B and height h.

Let planes parallel to the base of the pyramid divide the altitude into n equal parts and intersect the pyramid in n



similar sections B, B_2, B_3, . . . B_n, including the base. Upon these parallel plane sections as bases construct n triangular prisms each of height $\frac{h}{n}$ and all having edges parallel to any chosen edge of the pyramid.

The volume of the bottom prism will be $B\frac{h}{n}$, the volume of the next above it $B_2\frac{h}{n}$, of the next $B_3\frac{h}{n}$, etc. Designating by V' the sum of the volumes of the n prisms we have

$$V' = B\frac{h}{n} + B_2\frac{h}{n} + B_3\frac{h}{n} \quad . \quad . \quad . \quad + B_{n-1}\frac{h}{n} + B_n\frac{h}{n}$$

$$= B\frac{h}{n}\left[1 + \frac{B_2}{B} + \frac{B_3}{B} \quad . \quad . \quad . \quad + \frac{B_{n-1}}{B} + \frac{B_n}{B}\right]$$

But since the bases of the prisms are similar triangles their areas are proportional to the squares of their distances from the vertex of the pyramid. Therefore:

$$\frac{B_2}{B} = \left(\frac{n-1}{n}\right)^2\frac{h^2}{h^2} = \frac{(n-1)^2}{n^2}\ ; \qquad \frac{B_3}{B} = \frac{(n-2)^2}{n^2}\ ; \quad . \quad . \quad .$$

$$\frac{B_{n-1}}{B}=\frac{2^2}{n^2}\,;\quad \frac{B_n}{B}=\frac{1^2}{n^2}$$

Whence

$$V'=B\frac{h}{n}\left[1+\frac{(n-1)^2}{n^2}+\frac{(n-2)^2}{n^2}\ldots+\frac{2^2}{n^2}+\frac{1}{n^2}\right]$$

$$=\frac{Bh}{n^3}\left[1^2+2^2+3^2\ldots+n^2\right]=\frac{Bh}{6n^3}n(n+1)(2n+1)$$

$$(1)\quad V'=\frac{Bh}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

By the theorem of limits when two variables are always equal and approaching limits the limits are equal. Now the sum of the volumes of the prisms is a variable V' which approaches as a limit the volume of the pyramid V as n increases beyond limit or as $n \to \infty$. It is only necessary then to establish the limit of the right hand member of the equation (1) in order to derive an expression for the volume V of a triangular pyramid. Therefore

$$\lim_{n\to\infty} V' = V = \lim_{n\to\infty}\frac{Bh}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=$$

$$\frac{Bh}{6}(1+0)(2+0)$$

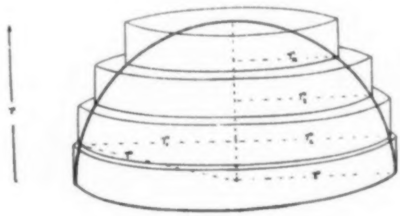
$$\text{or}\quad V=\frac{Bh}{3}.$$

The same procedure may be followed to derive the formula for the volume of a right circular cone by circumscribing cylinders. This exercise might well be left for the student to follow out for himself. Similar reasoning could then be applied to get the volume of any circular cone.

2. To find the volume V of a sphere of radius r.

Let planes passed parallel to the base of a hemisphere divide the radius perpendicular to the base into n equal parts and intersect the hemisphere in n circles of radii $r, r_2, r_3, \ldots r_n$,

including the base of the hemisphere. Upon these plane sections as bases construct right circular cylinders each of height $\frac{r}{n}$. The volume of the bottom cylinder will be $\pi r^2 \cdot \frac{r}{n}$, the volume of the next above it $\pi r_2{}^2 \cdot \frac{r}{n}$, of the next $\pi r_3{}^2 \cdot \frac{r}{n}$, etc.



Designating by V' the sum of the volumes of the n cylinders, we have

$$V' = \pi \frac{r}{n} \cdot r^2 + \pi \frac{r}{n} \cdot r_2{}^2 + \pi \frac{r}{n} \cdot r_3{}^2 \ldots + \pi \frac{r}{n} \cdot r_n{}^2$$

$$= \frac{\pi r}{n}[r^2 + r_2{}^2 + r_3{}^2 \ldots + r_n{}^2]$$

But from the figure

$$r_2{}^2 = r^2 - \frac{r^2}{n^2}, \quad r_3{}^2 = r^2 - (\frac{2r}{n})^2 \ldots r_n{}^2 = r^2 - (\frac{n-1}{n}r)^2.$$

Whence

$$V' = \frac{\pi r}{n}[r^2 + (r^2 - \frac{r^2}{n^2}) + (r^2 - \frac{2^2 r^2}{n^2}) \ldots + (r^2 - \frac{(n-1)^2}{n^2}r^2)]$$

$$= \frac{\pi r}{n}[nr^2 - \frac{r^2}{n^2}(1 + 2^2 \ldots (n-1)^2)]$$

$$= \frac{\pi r^3}{n}[n - \frac{(n-1)n(2n-1)}{6n^2}] = \pi r^3[1 - \frac{1}{6}(1 - \frac{1}{n})(2 - \frac{1}{n})]$$

Now the sum V' of the volumes of the n cylinders is a variable which approaches the volume $\frac{V}{2}$, of the hemisphere as n increases beyond bound. Therefore

$$\lim_{n\to\infty} V' = \frac{V}{2} = \lim_{n\to\infty} \pi r^3[1 - \frac{1}{6}(1 - \frac{1}{n})(2 - \frac{1}{n})]$$

or $$\frac{V}{2} = \pi r^3(1 - \frac{1}{3}) = \frac{2}{3}\pi r^3$$

and $$V = \frac{4}{3}\pi r^3.$$

3. To find the volume V of a frustum of a pyramid of lower base B, upper base b and height h.

Assume the pyramid completed and of height H. Let n prisms be escribed as before m of which lie above the upper base b. If the heights $H - h$ and H are incommensurable the ratio $\frac{m}{n}$ is a variable which approaches the limiting value $\frac{H-h}{H}$ as n and therefore m increase beyond bound.

Let $\frac{H-h}{H} = r$ then $H = \frac{h}{1-r}$ and $\frac{H-h}{H} = r = \sqrt{\frac{b}{B}}$

Designating by V' the difference between the sum of the volumes of the n prisms and the sum of the volumes of the upper m prisms; we have, as before

$$V' = \frac{HB}{n^3}[1 + 2^2 + 3^2 \ldots + n^2] - \frac{HB}{n^3}[1 + 2^2 + 3^2 \ldots + m^2]$$

$$= \frac{HB}{n^3}\ \frac{n(n+1)(2n+1)}{6} - \frac{HB}{n^3}\ \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{HB}{6}[(1 + \frac{1}{n})(2 + \frac{1}{n}) - \frac{m}{n}(\frac{m}{n} + \frac{1}{n})(\frac{2m}{n} + \frac{1}{n})]$$

As n approaches infinity, that is increases beyond bound, V' approaches V the volume of the frustum, $\frac{1}{n}$ approaches zero and $\frac{m}{n}$ approaches r. Therefore

$$\underset{n\to\infty}{\text{Limit}}\ V' = V = \underset{n\to\infty}{\text{Limit}}\ \frac{HB}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right.$$

$$\left.-\frac{m}{n}\left(\frac{m}{n}+\frac{1}{n}\right)\left(\frac{2m}{n}+\frac{1}{n}\right)\right]$$

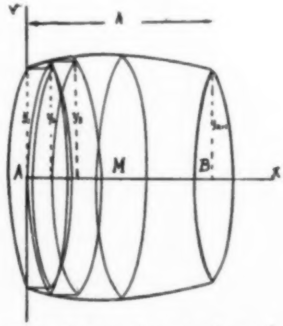
or $$V = \frac{HB}{3}(1-r^3) = \frac{Bh}{3}\,\frac{1-r^3}{1-r}, \quad \text{since} \quad H = \frac{h}{1-r},$$

and hence $$V = \frac{Bh}{3}(1+r+r^2) = \frac{Bh}{3}\left(1+\sqrt{\frac{b}{B}}+\frac{b}{B}\right) =$$

$$\frac{h}{3}(B+\sqrt{Bb}+b).$$

Graphs as interpretations of algebraic equations have come to play such a part in the teaching of secondary mathematics that it should be easy at this stage to demonstrate to the student the derivation of the prismoidal formula and explain for what surfaces of revolution it will be exact.

4. Suppose the curve whose equation is $y^2 = ax^2 + bx + c$ be revolved about the x-axis generating a surface including a solid of height h and having every section perpendicular to the x-axis a circle, to find the volume V.



Let the end sections or bases be circles of area A and B and the section midway between them a circle of area M. Let planes passed parallel to the bases divide the height h into n equal parts and intersect the surface in circles of radii $y_1, y_2 \ . \ . \ . \ y_n$.

Upon these plane sections as bases construct right circular cylinders each of height $\frac{h}{n}$. The volume of the first will be $\pi y_1{}^2 \cdot \frac{h}{n}$, of the second $\pi y_2{}^2 \cdot \frac{h}{n}$, etc.

Designating by V' the sum of the volumes of the n cylinders, we have

$$V' = \frac{\pi h}{n}[y_1{}^2 + y_2{}^2 + y_3{}^2 \quad . \quad . \quad . \quad + y_n{}^2]$$

Now, from the equation of the curve

$$y_1{}^2 = ax^2 + bx + c_{\,x=0} = c$$

$$y_2{}^2 = ax^2 + bx + c_{\,x=\frac{h}{n}} = a(\frac{h}{n})^2 + b(\frac{h}{n}) + c$$

$$y_3{}^2 = ax^2 + bx + c_{\,x=\frac{2h}{n}} = a(\frac{2h}{n})^2 + b(\frac{2h}{n}) + c$$

* * * * *

$$y_n{}^2 = ax^2 + bx + c_{\,x=\frac{n-1}{n}h} = a(\frac{n-1}{n}\cdot h)^2 + b(\frac{n-1}{n}\cdot h) + c$$

whence

$$V' = \frac{\pi h}{n}[ah^2\{\frac{1}{n^2} + \frac{4}{n^2} \quad . \quad . \quad . \quad + \frac{(n-1)^2}{n^2}\} + bh\{\frac{1}{n} + \frac{2}{n} \quad . \quad . \quad . \quad + \frac{(n-1)}{n}\} + nc]$$

$$= \frac{\pi h}{n}[\frac{ah^2}{n^2}\{1 + 2^2 \quad . \quad . \quad . \quad + (n-1)^2\} + \frac{bh}{n}\{1 + 2 \quad . \quad . \quad . \quad + (n-1)\} + nc]$$

$$V' = \frac{\pi h}{n}[\frac{ah^2}{n^2} \quad \frac{(n-1)n(2n-1)}{6} + \frac{bh}{n} \quad \frac{(n-1)(n)}{2} + nc]$$

Now the sum V' of the volumes of the cylinders is a variable which approaches the volume, V, of the solid generated as n increases beyond bound. Therefore

$$\lim_{n\to\infty} V' = V = \lim_{n\to\infty} \pi h\left[\frac{ah^2}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{bh}{2}\left(1-\frac{1}{n}\right) + c\right]$$

or $$V = \pi h\left[\frac{ah^2}{3}+\frac{bh}{2}+c\right] = \frac{\pi h}{6}[2ah^2+3bh+6c]$$

Now

$$A = \pi y_1{}^2 = \pi(ax^2+bx+c)_{x=0} = \pi c$$

$$M = \pi(ax^2+bx+c)_{x=\frac{h}{2}} = \pi\left(\frac{ah^2}{4}+\frac{bh}{2}+c\right)$$

$$B = \pi(ax^2+bx+c)_{x=h} = \pi(ah^2+bh+c)$$

whence

$$A+4M+B = \pi[2ah^2+3bh+6c]$$

so that

$$V = \frac{h}{6}[A+4M+B]$$

Since the equation $y^2 = ax^2+bx+c$ for different values of a, b and c may represent a straight line, a circle, a parabola, an ellipse or an hyperbola the formula derived gives correct values for the volumes of a cone, a sphere, a paraboloid of revolution, an ellipsoid of revolution, an hyperboloid of revolution or of portions of these between two parallel planes perpendicular to the axis of revolution.

For example to find the volume of an ellipsoid of axes $2a$, $2b$ and $2b$.

$A = 0 \qquad M = \pi b^2 \qquad B = 0 \qquad$ and $h = 2a \qquad$ giving

$$V = \frac{2a}{6}(0+4\pi b^2+0) = \frac{4}{3}\pi ab^2.$$

To find the volume, V, of a frustum of a cone of bases of radii r_1 and r_2 and height h

$$A = \pi r_1^2 \qquad M = \pi r^2 \qquad B = \pi r_2^2$$

in which $r = \frac{r_1 + r_2}{2}$ the radius of the midsection.

Then

$$V = \frac{h}{6}[\pi r_1^2 + 4\pi \frac{(r_1^2 + 2r_1r_2 + r_2^2)}{4} + \pi r_2^2] =$$

$$\frac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2).$$

To find the volume V, of a segment of a sphere the radii of the bases being r_1 and r_2 and the height h

$$A = \pi r_1^2 \qquad M = \pi r^2 \qquad B = \pi r_2^2$$

in which r is the radius of the midsection.
Let the radius of the sphere be R then

$$\sqrt{R^2 - r_1^2} - \sqrt{R^2 - r_2^2} = h$$

$$\sqrt{R^2 - r^2} = \frac{\sqrt{R^2 - r_1^2} + \sqrt{R^2 - r_2^2}}{2}$$

Multiplying the last equation by 2, squaring both, and adding we find

$$r^2 = \frac{h^2 + 2r_1^2 + 2r_2^2}{4},$$

so that

$$V = \frac{h}{6}[\pi r_1^2 + 4\pi \frac{(h^2 + 2r_1^2 + 2r_2^2)}{4} + \pi r_2^2]$$

$$V = \frac{\pi h}{2}(r_1^2 + r_2^2) + \frac{\pi h^3}{6}.$$

RECENT CHANGES IN THE TEACHING OF ALGEBRA[1]

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For a time I was undecided whether to make my title "Recent Changes in the Teaching of Algebra" or "Recent Improvements in the Teaching of Algebra." Finally I concluded that *changes* would be a safer word than *improvements* because we may not all think that the changes are improvements. I shall not discuss the content of ninth grade mathematics in general but shall limit myself chiefly to the algebra of the ninth grade, and shall, if time permits, consider two topics: changes in content and changes in classroom methods; that is, what to teach, and how to teach it. The Report of the National Committee and Prof. Thorndike's researches, as presented in his recent book, *The Psychology of Algebra*, are the two leading influences in changing our work in algebra. We have heard so many favorable comments on the work of the National Committee that I shall assume this afternoon the unpleasant task of calling attention to some of its unfavorable aspects. I am, so to speak, the counsel for the defendant; I shall plead the cause of that recently much abused reprobate, *Algebra*. It would be incorrect to say that the Report of the National Committee caused the changes in mathematics; rather, it summarized and called attention to work that had been going on for some time. Prof. Thorndike's book, on the other hand, not only comments on various methods that have been in use but also offers a great many new and interesting suggestions. At times Prof. Thorndike finds it necessary to make some remark like "This idea probably never even occurred to the generation of teachers in question" (page 97).

CHANGES IN CONTENT

On some matters regarding the content of first year algebra I think the National Committee was not so well informed as it might be. The report mentions eleven items that should be omitted from the seventh, eighth, and ninth grades. As I read these eleven items I wish you would estimate about how much

[1] Address, presented November 21, 1924, at the Annual High School Conference, University of Illinois.

time you have spent each year in teaching these particular topics. The list is:

Highest Common Factor and Lowest Common Multiple, except the simplest cases involved in the addition of simple fractions.

The theorems on proportions relating to alternation, inversion, composition, and division.

Literal equations, except such as appear in common formulas, including the derivation of formulas and of geometric relations, or to show how needless computations may be avoided.

Radicals, except as indicated in a previous section. (The section states that the consideration of radicals should be confined to transformations of the type $\sqrt{a^2b}$ and $\sqrt{a/b}$, and to the numerical evaluation of simple expressions involving the radical sign.)

Square root of polynomials.

Cube root.

Theory of exponents.

Simultaneous equations in more than two unknowns.

The binomial theorem.

Imaginary and complex numbers.

Radical equations except such as arise in dealing with elementary formulas.

When I read that list of "topics to be omitted" I wonder just how many teachers did teach (I say "did" because obviously we no longer do so since the National Committee advised us not to). I wonder just how many teachers did teach the theorems on proportion relating to alternation, inversion, composition, and division in the *ninth* grade? Undoubtedly many of us did spend some time on the square root of polynomials but how many of us taught cube root? Most of us have spent time on sets of equations in three unknowns, but how many of us taught the binomial theorem in the ninth grade? We may at times have spent a day or two on imaginary numbers, but did many of us discuss complex numbers in the ninth grade?

I think the Report of the National Committee compliments the ninth grade teachers immensely when it assumes that we were able to teach all these topics in the 36 or 38 weeks at our

disposal. I should like to pretend that I have been teaching the binomial theorem in the ninth grade but I must confess that I have never found time to teach even the simpler radical equations. I have never found time to spend even five minutes on the subject of Highest Common Factor. While I have managed to do some work in literal equations I cannot recall that I found time to discuss how literal equations serve to shorten needless computations, a thing that, to judge from the report, I should have been doing.

I call particular attention to these eleven topics because the National Committee states that *after we have omitted* these topics and decreased the amount of time on algebraic technique, we will have time for certain other topics, numerical trigonometry, for example. I believe that there are many teachers who omit all of these eleven topics and who decrease the amount of drill on technique as far as possible, and who still find that the work of the ninth grade must be planned very carefully in order to do justice to the remaining topics.

The College Entrance Board is undoubtedly well informed on what can be done in one year. Its first unit of credit in algebra has always borne the label "Algebra *to* Quadratics" in spite of fact that the study of quadratic equations is usually one of the topics included in the ninth grade. The new requirements of the College Entrance Board also omit quadratic equations but now include work on numerical trigonometry. I believe that the teacher who omits quadratic equations would find time to include some work on the trigonometry of the right triangle. In this part of the country, however, where the universities admit by certificate rather than by examination, the work on quadratics is a traditional part of the ninth grade and there is slight chance of its omission.

The College Entrance Board, like the National Committee, also suggests that some of the more difficult problems in the formal topics be omitted. Under *Factoring*, for example, it states:

"Only simple cases like

$$x^2 - 5x + 6 \quad \text{and} \quad x^2 + x - 2$$

are contemplated."

That statement would certainly be greeted with great applause by our pupils. Listen, however, to the next sentences in the report:

"A problem may, however, combine two or more of the cases here suggested, as in the polynomial

$$x^4 + 4x^3 - y^2 + 4x^2, \quad \text{or} \quad 2ax^2 - 4ax + 2a,$$

$$\text{or } x^4 + 4x^3 - x^2y^2 + 4x^2.$$

The teacher may wish to use such examples as

$$a^2 - b^2 + a - b \quad \text{or} \quad xy + 2x + 3y + 6$$

for purposes of instruction; but these cases are not included in the requirement."

I shall let you decide for yourselves whether or not we can decrease the time spent on factoring if "for purposes of instruction" we include such exercises.

In discussing the content of algebra Prof. Thorndike, page 223, enumerates fourteen points as the "abilities most worth acquiring in a year's course in algebra." They are:

(1) Ability to understand formulas.

(2) Ability to translate into a formula any clear statement of a relation.

(3) Ability to "evaluate" for any letter or other significant unit in formulas. (A list of 20 formulas is given to illustrate the degree of difficulty.)

(4) Ability to "solve for" or "change the subject to" any letter or other significant unit in such formulas as those just quoted. (This implies ability to solve a literal quadratic equation.)

(5) Ability to frame an equation or set of equations expressing any quantitative problem.

(6) Ability to solve such equations or set of equations if they are linear or quadratic.

(7) Ability to understand any graph representing by Cartesian coordinates the relation of one variable to another. Ability to make a graph for a table of values.

(8) Understanding of the elementary facts concerning the relations expressed by $y = ax$, $y = ax + b$, . . . $y = a/x$, . . . $y = ax^2 + bx + c$.

(9) Ability to solve for the constants in such equations, given the x, y values of two points on the curve.

(10) The abilities in algebraic computation required to deal with formulas and equations as stated above.

(11) Ability to understand and use negative and fractional exponents.

(12) Ability to use logarithms with multiplications, divisions, powers and roots.

(13) Such insight into the use of algebra for formulas expressing numerical relations as comes from the study of

(a) certain relations like $ax + bx = (a + b)x$, etc.

(b) certain formulas useful in computations or in understanding approximations, especially such as $(a^2 - b^2) = \ldots$ $(a + b)^2 = \ldots$

(c) the formulas for arithmetical progressions, geometrical progressions, and the binomial theorem.

(14) Certain informational abilities, especially the knowledge of the meaning of ratio, varies directly as, reciprocal, $\text{hypot}^2 = s_1^2 + s_2^2$, constant, variable, tangent, sine and cosine, the laws for corresponding dimensions in similar figures, and the use of tables of roots, powers, reciprocals, logarithms, tangents, sines, and cosines.

I am sure we all agree with Prof. Thorndike that these are the abilities most worth acquiring. The problem for us now is: How shall we manage our work so that these desirable abilities may be acquired *within the space of one year.*

There are in general three ways in which we may alter our present course. First, we may omit certain topics. We may omit cube root, the theorems on proportions dealing with alteration, inversion, composition, and division, imaginary and complex numbers, and on the days when you might have been teaching these topics, teach instead sines, cosines, and tangents. How you can *omit* what you have never taught is, to me at least, an unsolved mathematical problem. We may omit the binomial theorem as the National Committee suggests, and in its place study the formula for the binomial theorem as Prof. Thorndike suggests. We may, as the National Committee suggests, omit sets of three equations in three unknowns and in its place find, as Prof. Thorndike suggests, the three constants in the equation of a curve passing through three points.

A second method is that of beginning some of the work in algebra in the eighth grade. On the advisability of this method I am incompetent to speak as I have never taught in the eighth grade. We cannot assume that algebra *can* be taught in the eighth grade merely because we see some algebra in the eighth grade textbooks. We have been frequently told that the eighth grade work can be improved because most of it is a tiresome review. As long as so many pupils ended their education with the eighth grade, I believe that the eighth grade was the proper place for review work. And notice then that if in the future the ninth grade is to represent the end of many pupils' education, then there should be a great deal more review work in the ninth grade than we are now doing. The ninth grade must then do some of the work that the eighth grade has been doing in the past.

A third method of changing our ninth grade work consists in omitting what the National Committee calls "drill in algebraic technique." I believe that this is a most unfortunate phrase. Imagine a teacher of woodworking showing his class a hammer, and saying "We drive nails with it"; a saw, and saying "We cut wood with it"; an augur, and saying "We drill holes with it"; and then saying "We haven't time to practice with these tools; tomorrow we must do something practical—build a bookcase." Certainly drill in technique is the most essential item of all. What the National Committee very likely meant was that we should omit the more difficult and therefore, to their mind, the less practical exercises.

Let us consider for a moment what is a simple problem and what is a complex problem. The College Entrance Board, for example, states that in factoring quadratic expressions we need consider only the case where the coefficient of x^2 is 1. We would therefore use only expressions like $x^2 + 5x + 6$. After about five minutes of work on such expressions, what actually happens is this: I ask a pupil to factor $x^2 + 7x + 12$ and the pupil answers $(x + 3)(x + 4)$. The correct reply; but I venture to say that the pupil has not done any factoring. I might just as well have asked: "Guess two numbers whose product is 12 and whose sum is 7." The pupil was very likely unconscious of the process or meaning of factoring. A pupil who knows nothing about factoring can give just as good an answer after he

has heard a few other pupils recite. He does not think of factoring as the product of two quantities. He is thinking merely of the two numbers 3 and 4. The pupil does not begin to appreciate what factoring means until he reaches a problem in which the coefficient of x^2 is not 1, in which he must not merely guess at a pair of numbers but must actually find *all* the terms in the product. Factoring $x^2 + 7x + 12$ is a simple problem; in fact, it is so simple that a pupil cannot learn factoring by working at it. And the more such problems he works the more does the concept of factoring fade into the background.

We cannot state how difficult a problem is merely by looking at it. Prof. Thorndike calls attention to this idea on page 307, where he says:

"After a person knows algebra it is natural to think of a before $4a$ or $6a$, and of $a + a = 2a$ before $3a + 5a = 8a$. . . . To the learner $4a$ or $6a$ is easier to understand than a, and $3a + 5a = 8a$ is a better first problem than $a + a$. . . . It is an even chance that $3b$ times $4b$ is easier than $3b$ plus $4b$. In the former the child sees something happen to the bs as well as to the 3 and 4, while in the latter he must think them separately in some such fashion as 'something to do with 3 and 4 but nothing to do with the bs except annex one of them to the answer—where the other goes I don't know.' . . . Even few teachers remember to clear up his wonder as to what happens to the other b."

I point out these illustrations to show that when we try to decide what is a difficult problem and what is an easy problem, we must not judge by appearances. An exercise may be so simple that it fails to impart what we are trying to teach.

I have shown the objections to the three ways usually proposed for cutting down the work in algebra so as to make room for the new topics. And I have tried to show the unpleasant aspects of each method. The people on the other side of the case will now say, as we have doubtless heard them say frequently, "These represent the *ideals* for which we are striving. We do not expect to see them carried out next year or even in the next five years." Let me also, then, present some ideals, not quite so impossible in their attainment, a stepping stone, as it were, towards the greater ideals.

In planning this ideal I shall assume that it is better to teach a few topics thoroughly than to teach many topics half way. This keeps the pupil in a happier frame of mind because he sees that there is certain work that he is capable of doing. I assume also that we cannot introduce any of the new topics unless we omit more than the National Committee has suggested. My plan is based on an idea that Prof. Thorndike has stated better than I can state it. He says in several places (pages 232, 281) that we have frequently erred by carrying out the analogy between arithmetic and algebra too closely. He states on page 305:

"An order which is excellent as a means of arranging algebraic abilities for contemplation, or for keeping track of whether they are learned, may not be a good order in which to acquire them. The outline which we can survey in a few minutes is spread out over a year for the learner. We who know what is to come can use a system which is valueless to him who has yet to learn what is to come. What he most needs is a system and order that is good to learn by, not to look at."

We know, for example, that in the lower grades the order of the topics is always addition, subtraction, multiplication, and division, first with integers and then with fractions. Our textbooks in algebra have therefore assumed that the pupil should learn to add, subtract, multiply, and divide, first with integral polynomials and then with fractional polynomials. This is the traditional order: addition, subtraction, muliplication, division, factoring, and then the chapter on fractions in which the pupil studies again addition, subtraction, multiplication, and division, thus completing the four fundamental operations in a very logical manner before beginning any work on sets of equations or quadratic equations. An examination of the textbooks of thirty or forty years ago shows that in those days this very logical order was even more exact than it is today. We have broken away from it slightly by introducing the work on equations here and there between the four fundamental operations, and the indications are that in the future we shall break away from this order more and more.

My ideal is based on a new order of the topics, an order based on *the ease with which the pupil can learn them*. Very briefly, it is just this: put the chapter on fractions at the very end of the year's work. This order not only puts the hardest work at

the end of the year but it puts the most important work at the beginning of the year. Then the teacher who is courageous enough may omit some or all of the work on fractions and substitute for it some of the newer topics. The pupil who takes only one year of mathematics will never miss the work on fractions. The pupil who takes an additional year of mathematics, geometry, will not find his work in geometry hampered. The pupil who studies physics in his third year will do just as well without all the work on fractions. And the pupil who takes the third year algebra repeats most of the work on fractions anyway and is not handicapped.

The order of the topics for the year would then be as follows: we begin with the usual work on plus and minus numbers, formulas, equations, problems, as is usual for the first eight or ten weeks. The work on linear equations is followed by work on fractional equations with monomial denominators—not really fractional equations at all but merely linear equations with fractional coefficients. There can be as much work and review of arithmetical fractions as the teacher chooses because we can work with fractional coefficients when adding polynomials, we can multiply and divide monomials with fractional coefficients, and in the work on formulas we can use fractional values of the variables. This work on equations is followed by work on graphs and sets of equations. Ordinarily, sets of equations are treated in the second semester after the work on fractions. I place sets of equations in the first semester. The National Committee and Prof. Thorndike have presented statistics showing what are the most useful topics of algebra, the easiest for the pupil to learn, and the most interesting from the pupil's point of view. It is these topics that I have placed in the first semester's work. The second semester then begins with multiplication of polynomials and factoring, leading to the usual work on quadratic equations. Last of all comes the traditional work on fractions which, as I stated, can in part be omitted by the teachers who wish to introduce any new material. Most of us will agree that the chapter on fractions is the most embarassing topic in the first year's work. Also it is the least interesting to the pupil, the most formal in its technique, and the least important when we consider the future uses that the majority of the pupils will make of it.

The large amount of time that we now spend on fractions is due simply to the fact that we have carried over into algebra the desire to make a complete and logical presentation of the four fundamental operations of arithmetic regardless of their relative importance and regardless of the ease or difficulty with which the pupil can learn them.

CHANGES IN METHOD

In the remainder of the time I wish to speak of some of the changes in the presentation of particular topics. I shall not speak of intelligence tests, speed tests, hurdle tests, laboratory methods, supervised study and some of the other new methods, but wish briefly to call your attention to some of the suggestions that Prof. Thorndike has made.

At one time the postponement of the treatment of positive and negative numbers until the pupil had learned all the operations with positive literal numbers was thought helpful. Prof. Thorndike gives (page 316) four good reasons against this arrangement, and argues in favor of the traditional order. One reason is that at the beginning of the year there is on the part of most pupils a readiness to acquire new points of view, and it is good psychology to take advantage of this attitude of the pupil. Also, the pupil who learns algebraic addition, subtraction, etc., for positive numbers only finds it a disagreeable task to relearn the same operations with negative numbers.

Concerning the removal of parentheses, Prof. Thorndike points out that "numerous experimentors besides ourselves have found that making the general topic of removing parentheses subordinate to multiplication rather than to addition and subtraction decreases the difficulty." As you know, the traditional method states that if a parenthesis is preceded by a number -2, for example, we are to multiply the quantity by 2 and then subtract it and a parenthesis preceded by a minus sign meant that the quantity was to be subtracted. The new method means that a parenthesis preceded by -2 means that the quantity is to be multiplied by -2, and a parenthesis preceded by a minus sign means that the quantity is to be multiplied by -1, the number 1 being understood, as in x, meaning $1x$. My own classroom experience shows that this is a very good suggestion. And in

my own work I have gone one step farther in the use of negative numbers. In the solution of sets of equations we have what is called the method of addition or subtraction. In a problem where we would traditionally use the multipliers 3 and 7, for example, and then subtract the lower equation from the upper, I have found it useful to use the multipliers 3 and — 7 and then add the two equations. This method saves time because I need never refer to subtraction but merely to addition. Further, after telling the pupil at the beginning of the year how useful negative numbers are I see no reason why we should spend the remainder of the year trying to dodge them. I prefer to use them as often as possible so that the pupil sees how they do help to clarify our arithmetical processes.

In discussing the subject of graphs, Prof. Thorndike is particularly interesting. I believe that there are quite a few teachers who feel that the subject of graphs is not worth much time in the algebra class because it is an *easy* subject fit chiefly for the groups of low intelligence. Prof. Thorndike (page 73) states: "To read a statistical graph is within the power of the high school pupil untrained in algebra. But to make them offers such difficulty that only the exceptionally gifted child might be expected to master it without specific instructions. Still less would one anticipate that a critical attitude concerning the veracity of a graph and ability to detect falseness and graphic misstatement would develop without drill in that precise function."

And in another place (page 118):

"For pupils at the high school level, the erection of a series of columns and the formation of the curve joining the midpoints of their tops, seem adequate."

Considering the great amount of publicity which the *function* concept has received in the last few years, the following quotation (page 82) is of interest:

"For most high school students any larger use of the function concept than those suggested in the mathematical graph is of doubtful value. It is one of those fundamentally powerful conceptions whose elaboration has been one of the half dozen significant achievements of the race, but to the high school student it is vague and tantalizing and stimulating rather than clarifying."

Concerning our work with formulas, equations, and literal equations, Prof. Thorndike has made a very fine analysis showing how easily we can set up a great many conflicting ideas by our methods of teaching. In particular, our methods of teaching literal equations has been very confusing. The usual method is to study literal equations a little at a time. Some literal quadratics are studied after the usual quadratics; some literal linear equations are the usual linear equations; after sets of equations we have some sets of literal equations; and mixed with all of them are some formulas here and there. Prof. Thorndike states (page 131) that "the indiscriminate practice with what are now called literal equations would be replaced by two distinct lines of work. First there would be given, in connection with real formulas practice in expressing any one of the variables in terms of the others, that is, in solving for that variable. Second, there would be given, in connection with typical forms of relation lines, practice in understanding the meaning of the constants concerned as well as the meaning of the two variables."

I have mentioned briefly only a few of the suggestions in Prof. Thorndike's text, and have naturally spoken of those items which interest me most either because I agree with them or because I have tried them in my own classes. I know of no better way of improving the teaching of algebra than to read and study and try the many good suggestions found in this book.

A NEW TYPE FINAL GEOMETRY EXAMINATION

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The need for a final geometry examination of the newer type is apparent when one considers the inherent limitations of the essay form of test. In the usual examination paper, the pupil is expected to display his grasp of geometry by writing in full the proofs of three or four "book" propositions, by doing a simple computation example, by making one construction, and by solving three or four "originals." Some examinations, indeed, consist of but six questions, generally at least one of each type. An obvious disadvantage of the small number of problems is that they can cover only a very limited section of the subject matter of plane geometry. A pupil may be fortunate in happening to know these theorems particularly well. But, conversely, it is not inconceivable that his classmate may have a wider knowledge of the field and yet make a miserable showing on this particular part of it. Furthermore, the essay examination tends to stress only one side of geometric ability: the ability to *prove* a statement. Perhaps this is the most important of the skills geometry attempts to develop, but it surely is not the whole story.

Beside the limitation of giving only a restricted sampling of a pupil's skill and knowledge, the essay examination has another drawback: this is that the examinations of succeeding years are of varying degrees of difficulty. Now while the performance of pupils in geometry classes in a small school may shift from year to year due to the changing of teachers and the changing personnel of the class, it is unlikely that this should be the case when an examination is given to many classes in many schools. One would expect differences in the individual schools to neutralize each other. It happens, however, that on examinations given to many students in successive years, the proportion who make a certain standing changes from year to year. In the case of the College Entrance Examination Board's examination in plane geometry, less than half the candidates reached or exceeded the grade of 60 per cent in five of the years from 1910 to 1920 and more than half reached or exceeded it in the other

years.[1] The proportion varied in that time from below 35 per cent to about 65 per cent. Clearly on such tests a grade of 60 means little. In the first of the extreme cases here quoted, this grade places one in the top third of the candidates according to this examination; in the second case, it shows that one is not in the bottom third. (In justice to the Board's examinations it must be said that these extremes were not in successive years.)

Acting on the assumption that the examination is trustworthy, we sometimes resort to the scheme of shifting our scale of marking so that about the same per cent pass this year as passed last year. This, however, for the examinations of a few classes, is based on further hypothesis that this year's people are of the same calibre as last year's and in view of the smallness of the sample, this is untenable. With a larger number of classes, the device would be excellent were we sure that the examination itself were trustworthy. What would be ideal would be to have an examination of so many questions that the danger of being crammed to pass would be in part avoided, and to have a reserve of alternative problems whose difficulty is known. Then by a process of substitution, we might construct other examinations of approximately equal difficulty.

By the word *trustworthy* or *reliable*, one means the extent to which one examination ranks a group in the same order as does a similar examination. If the two tests rank the pupils in identically the same order, the reliability is said to be 1. If rank on one has no connection with rank on the other, the reliability is 0. The question is pertinent in regard to state or college entrance examinations for this reason: the less reliable an examination, the less likely it is that the lower group of those who pass this year would have passed had they had another examination of the same type. Conversely the upper group of those who failed might have passed the second examination.[2]

In these days of overcrowded freshman classes in college when for many Eastern institutions it is not simply a question of passing examinations but also of being in the top group of those

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[1] *Twentieth Annual Report of the College Entrance Examination Board*, 1920; p. 7.

[2] For a discussion of this in respect to the College Board see *The Reliability and Difficulty of the College Entrance Examination Board Examinations in Algebra and Geometry*. Ben D. Wood, C. E. E. B.; 1921.

who do pass, the importance of ranking a student properly cannot be exaggerated.

And what does *passing* mean? Is a 60, a 70, a 75 a shibboleth worth vastly more than a 59, 69, or 74? Wouldn't it be more desirable to know that such and such a score has been reached or exceeded by half the children tested? Or that so-and-so's record places him in the top ten percent of the children from many schools?

There are other mechanical details to be considered: the objectivity of scoring, the time needed to grade the examination, the accuracy with which it can be scored. Making a scoring scale for an essay examination is clearly arbitrary. But while the arbitrariness of the weighing of a given question of the new type examination must be admitted, we are less perplexed by the necessity of deciding what constitutes an acceptable reply. Where the essay examination distributes individuals according to the completeness of their replies, the new type examination does so according to their proportion of correct responses to a greater number of questions.

Considerations such as these have led to various attempts to build a final examination in plane geometry that would be in measure free from these drawbacks. One of these was begun by Mr. Raleigh Schorling, then of the Lincoln School and now of the University of Michigan, in 1921. In the spring of that year, experiments with his preliminary test showed it to be of value. Since then, three revisions of the test and the construction of a second form of the last of these revisions have been made by the writer. The third revision has been studied to determine its reliability and the distribution of its scores.

The original idea of the test was this: to make an examination which would be comprehensive in the ground covered, which would require a minimum of writing, which would be objective and therefore quick to score, and which could be taken in a short time.

In its original form,[1] the test was in seven divisions which may best be described by the type of question each contained. These were: completion sentences; true-false of conclusions from given data; true-false of converse statements; matching

[1] For a discussion of the original purpose of the test see *The Reorganization of Mathematics in Secondary Education;* pp. 390-395.

reasons against conclusions; drawing valid conclusions from given data; computation; analyzing constructions.

In order to get a measure of both the very good and the very poor pupils, it was planned that each part of the test should contain questions of graded difficulty and to keep the test within a short time limit, it was proposed that the questions of each unit be arranged in order of difficulty. Ideally, then, a pupil's progress through each unit should be an indication of his ability and if he failed on the fourth question of a group, he might reasonably be expected to fail on the later ones. Such a ladder-like arrangement permits one to limit the time on each section, for if a pupil meets an obstacle early in a part of the test, no amount of time would suffice to get him over the later difficulties of that group of questions.

One of the first things to be done to the original test was to expand each part to many more than the original number of questions and to try these out to get an idea of their relative difficulty and to gauge the value of each section. Table I shows the correlations between the rank given the pupils in a class by their teacher and the rank given them by the expanded sections of the test. These expanded sections took approximately 40 minutes each. The matching test had already been eliminated as being too easy.

The next step in the revision was to condense each part so that the whole test might go into an hour's time. As a result, the questions of the two forms of the test are arranged in an order that is nearer to the real order of difficulty than any one we might have decided upon by mere opinion. And we have in reserve questions of the same type from which other forms of the test might be constructed. Furthermore, each unit contains at least one question that will be answered correctly by 90 percent of the children who take this as a final examination in plane geometry, and each unit also has a question that 20 percent or less will do correctly.

The subject matter of the test is within the limits defined by the National Committee in its list of fundamental and subsidiary theorems and an effort has been made to restrict the terms and symbols to those of the committee's report.[1] Thanks

[1] *Reorganization of Mathematics in Secondary Education;* pp. 55-60 and 74-79.

to the helpful criticism of the work by a number of teachers, I believe that the geometry of the test can be justified as being common to the various arrangements of the subject.

The reasons for the selection of the types of questions were these: if the essence of geometry is demonstration and a consciousness of what constitutes a logical argument, we are interested in a child's information about geometric facts, in his ability to draw an inference from given data, in his ability to judge the validity of a conclusion, and, as a secondary matter, in his ability to analyze a construction and to use geometric facts in computation. This analysis is faulty in its omissions but I believe that the presence of each of these items can be justified.

Table I

CORRELATION BETWEEN TEACHER'S JUDGMENT AND RANK ON EXPANDED SECTIONS OF THE TEST

Number in Class	Completion Sentences	True-False of Conclusions	Converses	Drawing Conclusions	Computation	Construction
13	.80	.58	.81	.74	.72	
18[1]	.41	.24	-.31	.59	.34	
22		.78		.58	.64	.75
22		.64		.87	.79	.79
16		.68		.39	.71	.63
21		.44	.55	.46		.82
24		.35		.01	.44	.46
24	.20					
25	.60					
30	.49		.47		.71	
33			.28			.26
Median	.49	.58	.47	.58	.71	.69

The present form of the test differs in some respects from the original and as the reasons for the changes are illuminating

[1] It should be noted that the second class in this list is from a school where an unusual amount of time is given to plane geometry. This may account for the low correlations. The Completion Sentence test was also given to five other classes that took this test alone. The correlations are .94, .72, .90, .32, .57. The numbers in these classes were 9, 10, 11, 11, 14, respectively. These seemed to give the completion sentence value slightly above the converse statements.

I shall give them in some detail.[1] It should be remembered that changes in the order of questions have been made in each part of the tests and where the type of question remains unchanged the individual questions have been altered in many cases.

The first section contains completion sentences testing information about definitions, axioms, postulates, and a few theorems. The main question about this section is whether it tests knowledge or linguistic ability. Should further revision of the test be made, it might be wise to replace these completion sentences by multiple choice questions. For instance, instead of asking a pupil to fill in the blanks in the sentence: *The Greek letter π represents the..................of the..................of a circle to its............................*, we would ask him to underline the correct statement in the following: *The Greek letter π is a symbol which stands for the ratio of the circumference of a circle to its radius, the ratio of the circumference to the diameter, the ratio of the area to the diameter.*

The next part of the original test gave a statement about a diagram printed on the page and a conclusion supposedly based on the given facts. The problem was to tell whether this conclusion was true or false. But this neglected a fundamental fact. We are in the dark through lack of sufficient information quite as often as we are in a position to assert that a thing is or is not so. The test has been altered to tell whether the conclusion is *true*, or *false*, or whether *you cannot tell.*

The third part of the test asked you to read a statement, think of its converse and tell whether the converse was true or false. The first revision asked you to write the converse and check its truth or falsity. The result might have been foreseen. There were two possibilities for the check and three for the statement: the actual converse, a restatement of the theorem, or a hash of words. All six of the combinations appeared. Under these circumstances ten or twelve questions were too few to be used. At this point, one of the critics of the test pointed out the fact that the converse of a definition is true if the definition is a good one, and that the complete converse of a theorem in plane geom-

[1] Excerpts from the original form are given in thte National Committee's Report; pp. 391-394. The two forms of the revised test are published by the Bureau of Publications, Teachers College, New York City.

etry is never false. These considerations would have led to a radical change in that section of the test had the section seemed to be of sufficient value to warrant its retention.[1]

The fourth part of the test consisted of matching reasons against conclusions. This proved so easy that practically all the members of a class made perfect scores. The explanation was probably not in the device itself but in the fact that the number of reasons from which the choice was to be made was only four more than the number of conclusions to which they were to be matched. Furthermore, the difficulty would have been greatly increased had each statement been selected with a view to making a pupil think which one of a group of apparently plausible reasons would apply.

The sections where one draws inferences from given data, where one applies geometric formulae to work in computation, and where one analyses constructions are practically unchanged. In the latter case, we are open to the criticism that ability to analyze someone else's drawing does not imply ability to actually make the drawing *de novo*.

The correlations between the records on each part of the test with the total score on the two forms combined is given in Table II. Here the pupils' score on Form A Part I was added to his score in Form B Part I and the correlation between these sums and the sum of the total grades on Forms A and B was computed. The same process was carried out for each section.

Table II

CORRELATION BETWEEN THE COMPOSITE OF SCORES ON PARTS OF THE TWO FORMS OF THE TEST WITH THE TOTAL COMPOSITE SCORE ON THE TWO TESTS

140 Cases

Computation	.81
Drawing Conclusions	.73
Judging Conclusions	.68
Analyzing Constructions	.58
Completing Sentences	.55

[1] See Table I.

Table III shows the correlation between the rank given by each part of Form A and the rank given a pupil by his teacher.

Table III

CORRELATION BETWEEN RANK ON EACH PART OF FORM A WITH TEACHER'S JUDGMENT OF CLASS

School and Class	Number in Class	Completion Sentence	Drawing Conclusion	Judging Conclusion	Analyzing Constructions	Computation	Test A with Teachers Rank
A 1	26	.33	.41	.13	.41	.59	.52
2	36	.57	.56	.29	.40	.46	.76
3	25	.46	.45	.28	.35	.57	.46
4	30	.39	.38	.17	.16	.12	.56
5	23	-.04	.59	.62	.64	-.05	.67
B 1	30	.32	.67	.59	.52	.49	.63
2	22	.41	.88	.56	.60	.83	.81
D 1	27	.36	.75	.41	.52	.49	.72
Median		.375	.575	.35	.465	.475	.65

As to the mechanics of the test, the present time limits for each section make it possible to give the test within a sixty-minute period, but its make up is such that it can, if necessary, be given in two installments. It is the writer's present conviction that keeping the test to this short space of time has been a hindrance rather than an advantage. A longer test would probably have had greater reliability and might have been far more satisfactory.

The scoring is much more expeditious than the essay examinations. Using a key, the test may be scored at the rate of thirty to the hour.

But the real question is does this test measure ability in plane geometry? Is it reliable? What score should children make on it?

Our criteria for the first question are twofold: the opinion of the class teacher and the result of an essay examination. Accordingly each of the cooperating teachers was asked to rank his pupils before giving the examination. The correlations between class rank and test rank are given in Table IV.

Table IV

CORRELATION BETWEEN TEACHER'S JUDGMENT AND RANK ON FORM A

School	Class	Number in Class	Correlation Form A Teacher's Judgment
A	1	26	.54
	2	36	.76
	3	25	.46
	4	30	.56
	5	23	.67
B	1	32	.64
	2	22	.81
C	1	23	.73
	2	13	.72
	3	9	.84
	4	16	.52
D	1	27	.74
E	1	17	.67
	2	19	.75
F	1	34	.81
G	1	21	.73
	2	15	.47
	3	24	.88
	4	22	.83
H	1	47	.83
I	1	18	.67
	2	19	.75
	Total Cases	576	Median .735

A comparison was made between the two forms of this test and the New York State Regents examinations. The advantage of using the regents examination was that all the students of each class were obliged to take it as a final test. The results of this appear in Table V.

Table V[1]

CORRELATION BETWEEN TEACHER'S JUDGMENT, TEST MARKS ON FORM A AND FORM B, AND THE REGENTS EXAMINATION

School A Class	Number in Class	Teachers Rank and Form A	Teachers Rank and Form B	Teachers Rank and Regents	Regents and Form A	Regents and Form B
1	26	.54	.49	.59	.61	.60
2	36	.76	.71	.67	.67	.78
3	25	.46	.54	.43	.46	.55
4	30	.56	.59	.57	.64	.72
5	23	.67	.25	.52	.71	.57

The reliability as found by giving the two forms of the test to the same children appears in Table VI.

Table VI

THE RELIABILITY OF THE TWO FORMS OF THE TEST

School	Class	Number in Class	Teachers Rank and Form A	Teachers Rank and Form B	Form A with Form B	Test Given First
A	1	26	.54	.49	.77	A
	2	36	.76	.71	.77	
	3	25	.46	.54	.55	
	4	30	.56	.59	.79	
	5	23	.67	.25	.53	
B	1	30	.64	.80	.85	B
	2	22	.81	.85	.88	
C	1	23	.73	.55	.77	A
	2	13	.72	.93	.82	
	3	9	.84	.83	.81	
	4	16	.52	.54	.73	
D	1	27	.74	.41	.46	A

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[1] These results are in harmony with those obtained by the Commission on New Types of Examinations of the College Entrance Examination Board. See their report p. 13.

School records Hawkes-Wood Examination	School record College Board Examination	Hawkes-Wood College Board
.58	.54	.49

The correlation between the scores on the two forms of the test for 284 cases is .72. Corrected for attenuation, this becomes .83.

The scoring scheme has been to allow one point for each of the sixty questions of the test. To date, no one has made a perfect score on either form of the test and no one has made a zero one.

The distribution of the scores resembles a normal distribution curve.

Table VII

DISTRIBUTION OF SCORES

Score	Form A	Form B
0- 3.9	0	0
4- 7.9	1	0
8-11.9	2	0
12-15.9	13	3
16-19.9	50	14
20-23.9	73	22
24-27.9	114	42
28-31.9	135	81
32-35.9	117	46
36-39.9	91	40
40-43.9	63	27
44-47.9	22	9
48-51.9	17	4
52-55.9	4	1
56-59.9	3	1
60	0	0
Total	695	290
Q_1	25.0	27.5
M	30.6	31.2
Q_3	37.0	36.8

From Table VII, it would appear that Form B contains more questions of the easier grade than does Form A or that the practice effect for those who took both forms tended to increase the low scores as most of these students had Form A first, or it may be that the pupils who took the second form as well as the first were more highly selected than those who took the first form only.

This examination, then, distributes the children in approximately a normal curve, it has no undistributed scores at either end, it has a reasonable correlation with teachers ranks and fair reliability. It may be used to compare the performance in one class with that of the cases here given or to rank individuals against this group.

As a diagnosis of learning difficulties, it comes too late for the class, but not too late for the teacher. The study of the responses of 150 cases, taken at random, reveals some striking things. These come as a by-product of the test but they are significant and valuable.

Let us take the case of an axiom from the first part of the test. The problem is to complete this sentence: *If equals are subtracted from unequals, the remainders are............................in the..............................order.*

Of 150 cases,	113	say	*unequal* in the	*same order.*
	18		*equal*	*same.*
	15		*unequal*	*opposite.*
	1		*unequal*	*reverse.*
	1		*equal*	*opposite.*
	1		*equal*	*reverse.*
	1		*unequal*	*unequal.*

Three-fourths of them know how the axiom works, but how could anyone ever have two things equal in the *opposite* order?

The definition of π was successfully answered by 43.

18	*diameter* to *circumference.*
17	*circumference* to *radius.*
12	*radius* to *circumference.*
10 gave it as the ratio of	*area* to *circumference* or to *radius* or to *diameter.*
1	*radius* to *area.*
1	*radius* to *diameter.*

34 omitted it and 14 gave incomplete answers.

The salient point about these answers seems to be that the pupils recognize the elements involved but fail to know relations of these elements.

Another example that illustrates how a student may be satisfied with a half notion of what a thing is about, comes in the case of the definition of a square. To some, the important thing was equality of sides.

	31		*quadrilateral*	*with equal sides.*
	10		*figure*	
Of these people	4	said it was a	*polygon*	
	16		*parallelogram*	
	2		*rhombus*	
	2		*quadrilateral*	*with parallel sides.*

The equality of angles is more potent to the following:

6		*parallelogram*	*with equal angles.*
2		*polygon*	
3	called it a	*quadrilateral*	
4		*quadrilateral*	*with right angles.*
8		*parallelogram*	
1		*rectangle*	
16 omitted it.			

In the case of the computation examples, I think we will grant that the formula for the area of a triangle is one of the things that should be automatic. The problem was to find the area of a triangle whose base is 20″ and whose altitude is 6″.

2		30	
3	give	90	square inches
8		120	
1		70	

It is easy to see that 120 is obtained by taking the product of the base and altitude, 90 by taking half the base times the square of half the altitude, and 30 by taking half the product of half the base times the altitude. The 70 is probably a case of a poorly made 90.

The length of the altitude of an equilateral triangle is given less emphasis in plane geometry, yet the relation of the sides of a 30-60 right triangle are very necessary in trigonometry and this problem is closely allied to it and so it should be well known. Over half the pupils omitted the question of computing the altitude of an equilateral triangle whose side is 10″. Part of the answers given involved the square root of three in new and unexpected ways.

3	said	$25\sqrt{3}$
3		$\frac{10\sqrt{3}}{3}$
1		$2\sqrt{3}$
1		$15\sqrt{3}$

The square root of 5 also figures in the replies:

2	$\sqrt{5}$
3	$5\sqrt{5}$

Other answers are:

4	7.5
9	10
2	$6\frac{2}{3}$
1	20.8
1	50
4	8

Quoting these findings would be futile if they were not from classes we believe to have been taught at least as well as the

average for the country. More than that, the tests were taken seriously for they were either final examinations or tests just before finals. But these hodge podges of ideas make us question first whether what we are teaching should be taught or not, and, secondly, how we can avoid leaving the children in such a maze of misconceptions.

One use of such tests as this is, I believe, to give an idea of whether we are doing what we set out to do. We cannot be really doing it if in the end we find that children do not know fundamental axioms and definitions and that they cannot use theorems as tools. Tests, then, show some of the things we are not doing.

The future work with the test depends in part on our being able to get a better picture of a child's attainment than the judgment of his teachers or the essay examination. It appears to involve making a careful analysis of the skills we believe geometry should develop, and then making a study of the best methods of gauging these skills. Following this should come the construction of a number of tests on specific parts of the subject matter designed to test progress in each of these skills. These tests should be given at frequent intervals through the year, replacing in part, the regular class quizzes. The value of a final examination could then be measured by the ranking it gives a class in comparison with the composite of the year's tests and the part that each section of the final test plays in this analysis can be determined. We would then have a better basis for the inclusion of other sections in this final test and for the rejection of some which we now have. It might happen, for instance that we would find that we really do not need to introduce any section on solving originals in the test, the ability to do this *may* be measured indirectly elsewhere in the examination. On the other hand, we are quite likely to find that this is not the case and that some such work should be inserted. This series of tests is likely to be much more useful as a help in teaching than the final examination for whatever service it gives in a diagnostic way will come when this remedial work can be applied. To be more specific, I recently gave my own group a test which might be considered a beginning toward this series and I found a great confusion in their minds about definitions and postulates and theorems. Questions in class cor-

roborated this for the group showed magnificent courage in setting out to *prove* definitions. If geometry bears any relation to logic, this point was far more important than specific information about triangles, so, naturally, the next thing to do was to concentrate on this until the class appreciated it. In another group the same question might show that the class had already grasped the point.

The construction of such tests should not lead us to the point of believing that the sole objective of geometry teaching is the achieving of *norms* or the raising of medians. In so far as tests help us to discover teaching difficulties and to remedy teaching mistakes, they are valuable. And so long as the final examination is with us, it is incumbent upon us to make it as fair and as just an instrument as we can. This particular test is far from the best we might hope for, but one trusts it is a step in the right direction.

DRAWING FOR TEACHERS OF SOLID GEOMETRY

By JOHN W. BRADSHAW
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PART TWO

To meet the difficulty that one unaided picture does not suffice to reconstruct the space object, the engineer commonly avails himself of two or more pictures, plan, elevation, profile, which he places side by side. To discuss the mutual relations of these and to solve problems in space by means of constructions on these plane figures is the object of technical descriptive geometry as usually treated in the text books. This is really only a branch of descriptive geometry, which, broadly defined, includes every attempt to represent a space figure by a plane figure in which straight lines are represented by straight lines. We are here interested not in the method of the engineer but in another branch of descriptive geometry called axonometry. We can make out with one picture if we'll put into it something familiar, something that we shall agree represents a cube; or, what amounts to the same thing, if we'll introduce axes into the figure. We make use, then, of the frame-work of solid analytic geometry or coordinate geometry of space. Without presupposing any knowledge of this branch of mathematics, it is a simple matter to explain the fundamental notions on which it rests, as far as they are needed for our purpose.

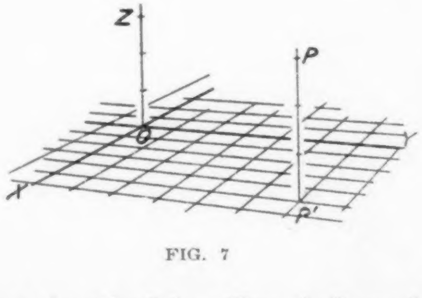

FIG. 7

The problem to be solved is: How shall we determine the position of a point in space, and how shall we convey this information by a figure? Imagine a fixed horizontal plane and in it a pair of rectangular axes OX and OY with equal units

marked upon them, a system of coordinates such as is used in plane analytic geometry (Fig. 7). From the point P, whose position we wish to describe, we let fall a perpendicular upon this plane and call the foot of the perpendicular P'. If then we give P' and the distance that P is above or below it, we shall know exactly where P is in space.

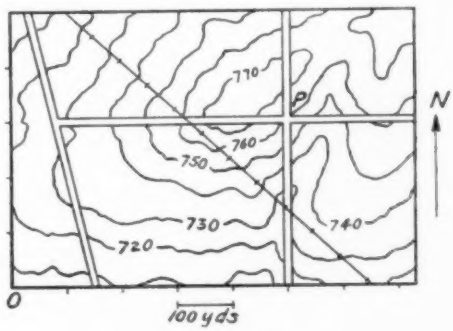

FIG. 8

Anyone who has used a contour map has really been applying this idea. The map (Fig. 8) is a picture of the horizontal plane and the contour lines serve to indicate the elevation of the earth's surface above sea level. Thus the cross-roads at P is 500 yards east and 300 yards north of O, and its elevation above sea level is 760 feet.

We are able to simplify matters somewhat if we can choose the same unit for measuring elevations that we use in the horizontal plane. Let z denote the number of these units in the line $P'P$, positive if P is above P', negative if below. We may then say that for every point P in space there is just one point P' and just one number z, and conversely, to every combination of a point P' and a number z there corresponds one and only one point P in space.

There is another way of expressing this determination of P that has the advantage of symmetry. Since it takes two numbers x and y to fix P', we may say that the point P in space is determined by three quantities x, y, and z. These are called the coordinates of P and we often speak of P as the point (x, y, z). Thus in Fig. 7 P is the point $(5, 6, 3)$. For those points

that lie in the horizontal plane we think of z as zero, so P' is the point (5, 6, 0). All those points for which x and y are zero lie on a vertical line extending up and down from the origin. This line is called the Z-axis, so in space we have three axes OX, OY, and OZ, each perpendicular to the other two. Every pair of these axes determines a plane: OX and OY, the horizontal XY-plane that we started with; OX and OZ, the XZ-

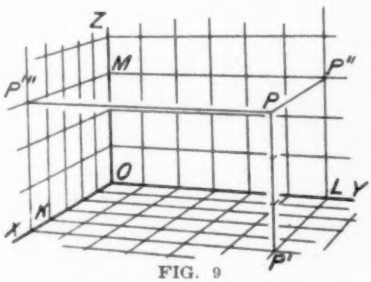

FIG. 9

plane; and OY and OZ, the YZ-plane. These three planes, each perpendicular to the other two, are called the coordinate planes. If from P (Fig 9) we let fall perpendiculars, PP'' on the YZ-plane and PP''' on the XZ-plane, it is clear that the first of these lines is x units long and the second is y units long. Hence we may think of the coordinates x, y, and z as all alike measuring the distances of P from three mutually perpendicular planes.

Still another way of looking at coordinates that is often convenient arises from a consideration of the planes determined by the perpendiculars. Let the plane $P'PP'''$ cut the X-axis in K, $P'PP''$ the Y-axis in L, and $P''PP'''$ the Z-axis in M. Then clearly $OK = PP''$, $OL = PP'''$, and $OM = PP'$. The point P appears as a corner of a rectangular parallelepiped bounded by the coordinate planes and planes parallel to them, and three non-parallel edges of this parallelepiped fix the coordinates of P. Perhaps the most frequently convenient set of edges to use forms a broken line by which P may be reached from the origin, as, for example, $OKP'P$ or $OLP''P$.

The method of axonometry is to associate with the space figure a system of axes which shall also appear in the drawing and give us something known for comparison; we must first study,

then, the picture of a set of axes. Now there is a large measure of freedom in setting up our apparatus for making pictures. After the axes have been chosen in some simple relation to the object, usually so that one or more will coincide with some of its important lines, we may place our picture-plane where we please and select any direction for the direction of projection. The simplest choice for the former is parallel to one of the coordinate planes, because then whatever lies in that plane appears in its true form and size. This means that two of the axes are pictured by perpendicular lines and that two of the coordinates of a point can be measured directly from the picture. The other axis will be perpendicular to the picture-plane and the position of its picture will be dependent upon the direction of projection.

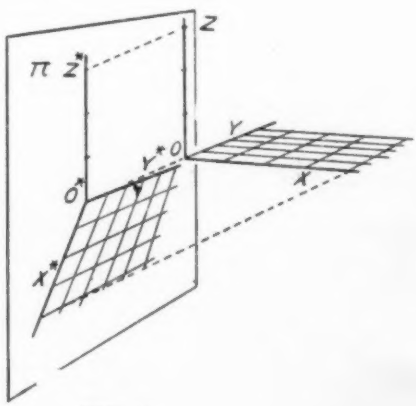

FIG. 10

In Fig. 10 we show the picture-plane parallel to YOZ, and have used the letters X, Y, and Z to denote points at equal distances from O, that is OX, OY, and OZ shall be understood to represent the units on the axes, and the direction of projection has been chosen so that the picture of OX falls in the angle between the negative axes OY and OZ and is somewhat shorter than OX *itself*. In Fig. 11 we have turned the drawing-board around so as to look at the picture directly. Here is a typical picture of axes as used in what is called oblique parallel projection or clinographic axonometry. One further simplification is common. Since after fixing the direction of projection, if the

drawing-board is shifted parallel to itself no change will be produced in the form of the picture, it is convenient to think of the picture-plane as not merely parallel but actually coinciding

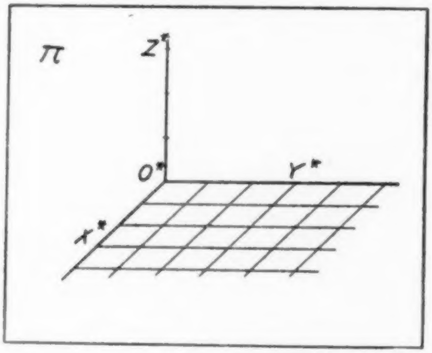

FIG. 11

with the YZ-plane. We say we are drawing in the plane YOZ and every point in that plane has a picture that coincides with itself.

Now a word about notation. A few conventions will materially assist in our further work. We shall agree to denote a point in space by an italic capital P, its picture by the same letter with a star P^*, the foot of the perpendicular from P on XOY by P' and its picture by P'^*; similarly an italic small letter a will denote a line, a^*, its picture; planes we shall designate by Greek letters. Pictures of points in YOZ, since they coincide with their originals, do not carry stars.

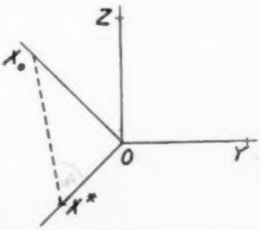

FIG. 12

From now on we shall interpret the typical picture of axes OX^*, OY, and OZ (Fig. 12) as implying that the Y- and Z-axes

lie in the plane of the drawing, that OX is a segment perpendicular to this plane equal in length to OY and OZ, and that the direction of projection is such that X is projected into X^*. Since the X-axis, being perpendicular to the picture-plane, is perpendicular to every line of that plane, the segment OX in space, its picture OX^*, and the projecting ray XX^* form a right-triangle, right-angled at O. The angle XX^*O of this triangle is the same as that made by every projecting ray with the picture plane, since all such rays are parallel. We shall denote it by w.

We may gain a clear idea of the form of this triangle and even measure the angle w by revolving the triangle about that side X^*O that lies in the picture plane until it falls into that plane. To construct it in this revolved position we have only to draw OX_0 perpendicular to X^*O and equal in length to OY and then to connect X_0 with X^*.

A model of a projecting ray may readily be constructed from Fig. 12. Cut the paper along OX_0 and X_0X^* and fold the triangle OX_0X^* along OX^* until it stands out perpendicular to the page. Then OX_0 will coincide with the X-axis and X_0X^* with the projecting ray XX^*. One should view the picture from a considerable distance in the direction of this line if one is to get the best possible impression from it.

It appears from this construction that instead of directly choosing a direction of projection we may choose a position for X^* and that will determine the direction of projection. Any point in the plane may be so chosen, but for practical reasons our choice is limited. Since a picture will make a satisfactory impression on the eye only if the direction of vision approximates the direction of projection, and since we usually view pictures from nearly in front, it is desirable to make the angle w large and choose OX^* correspondingly small. Sometimes OX^* is taken equal to the unit OX that it represents, then w is $45°$; but usually it is made shorter, $2/3$ or $1/2$ of OX. The important thing to notice is that with the conventions we have made about the segments OX^*, OY, and OZ placed as in Fig. 11, the arrangement of drawing-board and axes and the direction of projection are fully expressed.

It might seem that the best angle to choose for w would be $90°$. The disadvantage of this is that the X-axis would then

project into a point and it would be impossible to obtain from the picture distances measured parallel to this axis. When one is using two or more pictures, as the engineer does, this special sort of parallel projection called orthogonal, in which the direction of projection is perpendicular to the picture-plane, is very convenient.

This general statement of procedure will gain in clearness as we apply it to the drawing of the space figure represented by Fig. 15 and described as a right prism having a base that is a rhombus $ABCD$ with a 60° angle at A. We shall choose our

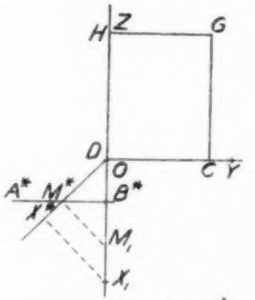

FIG. 13

origin O at D, the Y-axis coinciding with DC and the Z-axis with DH. We are drawing then in the plane of the face $DHGC$. We mark off our units OY and OZ (Fig. 13) and think of an equal unit marked off on the X-axis perpendicular to the drawing-board. Now we must decide on X^* and mark it in our drawing.

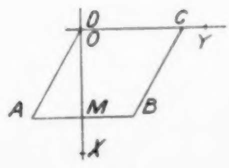

FIG. 14

Since the base of our prism is a 60°-rhombus (Fig. 14), our X-axis perpendicular to DC cuts AB in its midpoint M. The solution of our problem all hangs on locating M^*, the picture of M. Returning to Fig. 13, let us think of revolving the base of our figure about DC until it coincides with the picture-plane.

The X-axis will then fall on the continuation of the Z-axis and X will fall at X_1 so that $OX_1 = OX = OY$, and M will fall at M_1 so that $OM_1 = DM$. Join X_1 with X^*. An interpretation of this line X_1X^* contains the solution of our problem.

If we think of the motion of the point X as it goes to X_1, it traces out the quadrant of a circle in a plane perpendicular to the Y-axis, the center of the circle being on that axis, namely at O. The segment X^*X_1 is the picture of the chord XX_1 of this quadrant. Now every other point in the XY-plane, in particular the point M, also traces out a quadrant of a circle, and the chords of all these quadrants are parallel and hence have parallel pictures. The picture M^* is then determined by two conditions, it must lie on the picture of the X-axis and M_1M^* must be parallel to X_1X^*. The next step is to draw A^*B^*. Since AB is equal and parallel to DC, A^*B^* must be also, and it must have M^* as its midpoint. To complete the base we have only to draw A^*D and B^*C. Then A^*B^*CD is the picture of the rhombus in the XY-plane.

The vertical edges of our prism cause no trouble. Being parallel to the Z-axis, their pictures are also parallel to that axis, and these edges appear in their true length since they are parallel to the picture-plane. If we connect the upper ends of these edges, we obtain the picture E^*F^*GH of the upper base congruent with the picture of the lower.

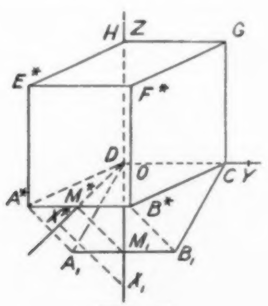

FIG. 15

In actually carrying out the steps we have described it is a matter of convenience to draw the base of the prism as it would appear in revolved position in the YZ-plane, that is to combine Fig. 13 and Fig. 14 into one figure. This we have done

in Fig. 15. The advantage would be more noticeable in a less simple figure, but even here certain checks on the accuracy of the figure appear that are not without value. For instance, when the base is revolved into the YZ-plane A and B trace out quadrants like the path of M. The pictures of the chords of these quadrants A_1A^* and B_1B^* should therefore be parallel to M_1M^*.

The reader is strongly urged to reproduce these figures for himself. It is only by drawing that one can learn either to draw or to understand drawings.

One is constantly required in this work to draw through a given point a parallel to a given line. The draftsman's method of solving this problem by sliding one triangle upon another is rapid and accurate.

In our next article we shall show how to determine the direction of projection that must have been used in drawing Fig. 1 of the Board's Document.

WHAT THE TESTS DO NOT TEST

By HELEN M. WALKER
University of Kansas

The question as to the adequacy of our standard tests of mathematics is only a very modern revision of the ancient and honorable query as to the aims of mathematical education, a subject upon which far wiser people than I have disagreed. It is evident at once that we cannot consistently have one set of goals operative while we teach and another while we test the results of that teaching; we cannot teach mathematics for its character content and test only for mechanical skills. It is of course easier to write a test which will measure dexterity in manipulating fractional equations than one which will measure the "honesty of thinking" about which most of us talk upon occasion, yet if we are to maintain the integrity of our own thinking we must face this dilemma.

I write as one who has been an ardent enthusiast for the testing movement, who is still deeply appreciative of the assistance which standard tests may give to the teacher who uses them intelligently, and who dares to dream that tests may yet be developed which will help us measure the contributions of mathematics to the growth of character, of standards of citizenship, of attitudes of mind. In the meantime I am aware that there is great danger that the progressive teacher of mathematics who has the vision to plan courses which shall really contribute to the life needs of students may not have the time or the psychological technique necessary for writing tests, and that the psychologist and professional educator who has the technique to write tests may not have the specialized knowledge of the subject to enable him to emphasize those aspects which should be made increasingly prominent. Some of the best known mathematics tests now available were written to fit courses as the authors found them and so emphasize features which many of us feel should be eliminated from our high school courses. If the tests are to be written to measure proficiency in the courses as they stand at the present time, and if teachers are to make it their main objective to train classes which will score high on the tests so written, then we are faced

with the possibility of having our present courses crystallized—or rather verily embalmed in the status quo—just at the time when mathematicians, psychologists, and professional educators are working together to put more vitality into secondary school mathematics.

Those of us who have always clung to the conviction that mathematics properly taught might influence one's habits of thinking, are in these last few years breathing a deep sigh of relief because the psychologists have modified their position enough that it is once more respectable to speak of character values we may hope to achieve through teaching mathematics. During the time that the psychologists denied the possibility of any transfer at all, it was inevitable that the emphasis in teaching mathematics should be upon the informational side: we should teach algebra chiefly for its value to the sciences, and geometry as a field in which to develop spatial perceptions, the appreciation of geometric form in nature, art and industry, the acquisition of those geometric concepts encountered in general reading, and some acquaintance with elementary design, surveying, and the like "practical" matters. Now the pendulum is beginning to swing back again, and teachers are once more talking about the intangibles, they once more dare to suggest that mathematics may contribute some essential things toward the development of a finer human life. On the other hand we no longer are so superstitious as to believe that we can teach the mechanical side of mathematics and reap good citizenship accidentally. The process is reversed. Now we analyze the student mind to see what habits of thought are desirable and need training, and then planning definitely for those habits we become convinced that mathematics offers an excellent medium for training some of them. Let us then agree that we may get out of a course in mathematics almost any "life values" for which we definitely plan, and that unless we do consciously work for them such character values will be negligible. We are again faced with the necessity of finding out what we are teaching it for, a most pertinent inquiry.

Among the subjects usually treated in a first course in algebra, the formula in particular offers an opportunity to give the student a new reverence for the harmony of the universe.

When we introduce a student to the formula, why do we not permit him to see in it an expression of one of the timeless laws of creation? $s = \frac{1}{2}gt^2$ is not a product of the evolutionary process, rather it has been one of the divine agents in working out that process. Let the student know that scientists are spending their lives trying to discover and formulate the laws of the universe, to "think God's thoughts after Him." In the physical sciences they have been able to see many relationships so clearly that they have set them down in algebraic language. In other fields such as psychology and the social sciences, most laws or tendencies are not yet so fully understood that they can be recorded in algebraic symbolism, though that is the goal toward which we strive. Let the student see the formula as a shining ideal of clarity, and let him make formulas to express his own ideas and discoveries. Most of the work with formulas has consisted solely of mechanical drill in changing the subject of the formula. We have given the student long lists of formulas drawn from the fields of physics, chemistry, electricity, and mechanics, all quite meaningless to him, and have said, "Solve this for s_0, solve this for t_n" and we have too often forgotten to add, "These formulas are taken from sciences about which you probably know very little. Here appears one of the great powers of algebra. Knowing nothing at all about electricity, you can take a formula which an electrician assures you is true, and by your knowledge of algebra can perform changes upon it and come out with a new formula which the electrician might never have discovered if he knew no algebra." There are two tests which are intended to measure grasp of the formula. One of these is purely an equation test with the addition of a single problem concerning the formula for the area of a triangle and the other has to do entirely with the mechanical technique of solving formulas. Neither one tests the ability to make and to interpret formulas, neither one measures a pupil's understanding of the immortality of the relations which formulas represent or their significance in the developing thought of science. To supplement these we do need new formula tests made on quite a different basis.

Another function of a mathematics course not referred to in any test except the Kelly Mathematical Values Test, which can

scarcely be termed a *standard* test, is to orient a student as to his position in the developing thought life of civilization. Shall we let him feel that mathematics sprang full-armed from the head of Zeus, or teach him that our number system, our terminology, our geometric knowledge grew with growing civilization, that other men have labored and we are entered into their labors, that we are "heirs of the past and trustees of the generations yet unborn"? Doubtless when all mathematics teachers make the history of mathematics an illuminating part of their instruction, tests will be written to measure their success At present we have none.

Without doubt one of the most important principles to be observed in the pedagogy of algebra is that an idea should always be presented ahead of the symbolism in which it is to be cloaked, the pupil should learn to use the algebraic language as a means of self-expression. Unfortunately it is quite possible for children to learn to manipulate the symbolism blindly much as a jack-daw learns to put words together without any comprehension that those words form a language and may be used to express thought. Some of the elements of good teaching as found in an algebra class bear striking resemblance to good teaching in foreign language classes. There is the same attempt to clothe thought with language, the same necessity for translating thought from one language to another—from the language of ordinary conversation into the concise and exacting language of algebra, and vice versa. If a pupil who glibly states that $11a+9a=20a$ takes some time to conclude that $11\times17+9\times17=187+153=340$ it is a fairly good indication that he had learned to manipulate symbolism rather than to think in the algebraic language. So far as I know there is no test which will give us any adequate and objective record of what the symbolism means to a student. We very much need tests which will help us discover the mathematical ideas in a pupil's mind apart from his ability to use the highly specialized language of algebra, and tests which will measure his success in translating those ideas into symbolic language. Here is a large area which needs exploration, and satisfactory tests would contribute much to the effectiveness of our teaching.

A few years ago The New York Times propounded this definition: "An educated man is a man who knows when a thing is proved." That ability to know what constitutes a proof involves, among other things, these factors:

1. A problem-solving attitude of mind.
2. A critical attitude toward data.
3. Skill in analysis.
4. Ability to generalize.
5. An understanding of the converse relationship.
6. Recognition of the place of assumptions in all thinking.

That each of these abilities is fundamental to vigorous thinking of any kind is at once apparent. Almost daily in our personal conversation we are annoyed by illogical people who mistake a statement for its converse. An understanding of the converse relationship is essential to clarity of thinking and can be taught in a geometry class more easily than anywhere else, yet often we are in such haste to amass geometric information that we do not give this truly important matter the attention it merits. (Here is a golden opportunity to show the pupil how the type of thinking done in mathematics should permeate all his life relationships.) We need more material from non-geometric sources to go hand in hand with our geometric material. Tests to measure success in this field are not now on the market, so far as I know, but can be written so easily that they should soon be available.

Tests of the ability to generalize are beginning to appear and it behooves us as teachers of geometry to lay stress on this aspect of our work. The head of the Department of English in the University of Kansas High School, Mrs. Louise Macdonald, has given most generous co-operation to the mathematics department, even visiting the mathematics classes to study the language difficulties of pupils there. Surely the attempt to formulate a theorem, to describe a geometric construction in clear concise English, or to phrase a question which will clarify the thinking of another pupil present language difficulties of an exacting sort. In this school the mathematics teaching is largely inductive, and the geometry students habitually formulate their own theorem after completing a proof. One day Mrs. Macdonald heard a lad sum up the results of a demonstration thus: "Therefore if a triangle has $AB=BC$, then angle A=angle C."

The teacher pointed out to him that information put up in so specific a form could not easily be used again; if the knowledge just gained were to serve him again and again in the future he must generalize it. Eventually the class worked out the theorem: "If two sides of a triangle are equal, then the angles opposite these sides are also equal." Within a week after this incident, this surprising sequel occurred in one of Mrs. Macdonald's own classes. They were reading Silas Marner and one of the boys offered as his explanation, "Silas was accused of taking some money he hadn't stolen, so he just ran away. But it didn't do him any good." At once a girl who had been in the geometry class spoke up, "That isn't the way to put it. You can't use this knowledge anywhere else if you make it a special case. You have to say it this way 'When one is falsely accused of committing a crime, it is useless to try to escape by running away.'" Had not Mrs. Macdonald made that particular visit to a geometry class, no one would have surmised that this observation was a transfer from a mathematics course.

In his recent book "The Mind in the Making," James Harvey Robinson says, "I am not advocating any particular method of treating human affairs, but rather such a general frame of mind, such a critical, open-minded attitude, as has hitherto been but sparsely developed among those who aspire to be men's guides. . . . Unless we wish to see a recurrence of . . . calamity, we must . . . create a new and unprecedented attitude of mind to meet the new and unprecedented conditions which confront us." The attitude of mind for which he pleads is one which will fearlessly examine all the assumptions on which its conclusions rest, whether those conclusions are intellectual, political, ethical, or religious. Such courage is rare and costly and seldom mentioned in the educational process. Has mathematics anything of value to offer toward the solution of this which is fast becoming one of the chief problems facing our civilization? If so, what standard tests are available to check up our success in the undertaking? Alas, none! The Kelly Mathematical Values Test already referred to, is a most stimulating and suggestive piece of work in this direction, but its results are not in any sense standardized.

It is a common complaint that all of us are gullible. Doubtless we shall continue gullible in spite of all attempts at reform, but the condition might be somewhat alleviated if boys and girls could be made to take pride in examining the basis on which their conclusions rest. Can we not help them to know whether a thing has been proved or not, to know when a campaign orator has made a sound argument and when he has given a mere opiate of high flung phrases about liberty and equality, and to insist that people shall not only draw correct conclusions but shall look critically at the assumptions out of which those conclusions grow?

Most of us have, packed away in the dark corners of our minds, assumptions which we have not the courage to bring out to the light of day and examine. These prejudices may be invested with a sanctity which makes it difficult for us to see them as hypotheses. The lay mind has a peculiar superstition to the effect that mathematicians prove everything, that they take nothing for granted, and thus there comes to be an unmerited opprobrium attached to the word assumption. We who recognize the important and honorable place of assumptions in mathematical thinking might do a great service to mankind if we could, through our geometry classes, train up a generation of boys and girls who could understand the place of assumptions in all thinking. Many texts still speak of axioms as "self-evident facts!" They are anything but that! Axioms are agreements made without proof but for the sake of proof. They may not even be true—we do not know that the parallel postulate is true, but so long as we agree to it, it is binding upon us, and determines the kind of geometry we will arrive at. Others of the usual list of axioms are certainly not true when we get outside of finite space. It is not the *truth* of the axiom but our *acceptance* of it which makes it binding. This it is which Dr. Cassius Jackson Keyser speaks of as "logical fate and intellectual freedom," freedom to chose our hypotheses and absolute bondage to these hypotheses when chosen. If we choose another set of hypotheses we elect another destiny, but to change our destiny, or the conclusion of our argument, we must go all the way back to revise our hypotheses. Herein also lies the opportunity to effect that honesty of thinking about

which most of us have had a good deal to say. Of course we must take the pupil into our confidence, must tell him frankly that we are trying to show him how to reason honestly and fairly in life situations, and that we hope to help him to better habits of thinking. Also we must be clear in our own minds as to what happens when people rationalize. Is it not that unconsciously they are harboring such unworthy hypotheses as: "I never make mistakes; I have a right to anything I intensely desire; anyone who criticizes me is stupid and cruel; people who come from a distant part of the world are certainly queer and probably bad; people who have a great deal more money than I have are probably dishonest; people who have much less money than I are probably shiftless." A frank examination of the hypotheses out of which we build our daily lives would leave the best of us little basis for self-congratulation.

Naturally you ask if I have any proof that the study of mathematics may be so utilized to transform attitudes of mind. If I am to maintain my own self-respect in the light of what has just been said about intellectual honesty, I must admit that I have no proof as we understand proof. I have only a working hypothesis and am making some experiments upon that basis.

The field is almost untried and utterly *untested*. He who would write tests for this sort of thing must have thorough training in both mathematics and psychology, neither alone is adequate. The task will not be easy but it offers a great chance for service.

NEWS NOTES

The sixth annual meeting of the National Council of Teachers of Mathematics will be held in Cincinnati, February 28, 1925. The Executive Committee and the President, Raleigh Schorling, have arranged a program that will be of unusual interest to teachers of mathematics. Special railroad rates may be had by members of the National Educational Association. Organizations of teachers are urged to send representatives to this meeting. Important business will be transacted. Make your plans now to attend this meeting. Note the program in this issue of the Teacher.

There are fourteen teachers in the mathematics department of the Oak Park (Illinois) High School. All are members of the National Council of Teachers of Mathematics. Mr. C. M. Austin, the head of the department, was the first president of the National Council.

In the article "Making Mathematics Interesting," by Augusta Barnes, the name of the publisher of Miller's *Romance in Science* was incorrectly given. *Romance in Science* is published by Stratford & Co., Boston.

The Club of the Men Teachers of Mathematics of New York City met for dinner and discussion at the Faculty Club of Columbia University on October 17th at the call of the chairman, Dr. J. R. Clark. Prof. W. B. Fite of Columbia gave a very instructive and interesting discussion of "Infinity and Infinitesimal." The following motion was unanimously adopted by the club: "It is the sense of this club that the College Entrance Examination Board should not publish an official syllabus in geometry but should judge the ability of a candidate in relation to the syllabus which he has used in his preparation."

—H. F. Hart.

The program of the mathematics section of the Western Division of the New York State Teachers' Association, held in Buffalo, November 13th and 14th, included:

1. A Study of the Report of the National Committee on Mechanical Requirements.
   - Chapter I. William David Reeve, Teachers College.
   - Chapter IX. John Greenwood, Technical High School, Buffalo, N. Y.
2. Guidance through Mathematics.
   - Mary E. Crofts, Masten Park High School, Buffalo, N. Y.
   - Chapter XIII. Emmons Ferrar, Hutchinson High School, Buffalo, N. Y.
   - Chapter X. Marien Ploss, Lafayette High School.
   - Chapter VII. Louis R. Witt, Niagara Falls, High School.
   - Chapter III. Raymond E. Brutsman, Principal Junior High School No. 4, Olean, N. Y.
3. Group Conferences on Class Room Procedure.
   - Geometry.
     Leader, H. M. Lufkin, Dunkirk High School.
   - Elementary Algebra.
     Leader, Harriet Bull, Masten Park High School, Buffalo, N. Y.
   - Intermediate Algebra and Advanced Mathematics, Room 15.
     Leader, George L. Lowry, Olean High School.

   The Program Committee recommends that each school appoint a delegate for each group who will take part in the discussion. The name of this delegate should be sent to the group leader.

   A member of each group will discuss the relation of the topic of that group to character building.
4. Reports of Group Leaders.
   Character Building: characteristics which can be taught in Algebra and Geometry.
5. Discussion.
   - (1) Should Algebra and Geometry be required as at present?
     (a) Both, (b) Algebra only, (c) One only.
   - (2) Should present method of ninth year Algebra and tenth year Geometry be continued?
   - (3) Should some kind of General Mathematics be introduced?
6. Question Box.
   Conducted by F. Eugene Seymour.
   Questions should be handed to the Chairman in writing on Thursday.

The elaborate and crucial experiment now being conducted near Clearing, Ill., by Prof. Albert A. Michelson, head of the Department of Physics at the University of Chicago, in collaboration with Prof. Henry G. Gale, of the same department,

is expected to demonstrate the effect of the earth's rotation on the velocity of light.

An earlier experiment conducted this summer by Professor Michelson in California had to do with determining more exactly the actual speed of light. The method consisted essentially in sending a beam of light from one mountain peak to another at known distance, reflecting it back from a mirror there, and timing the round trip. The sending station was located on Mount Wilson and the receiving and reflecting station on the top of Mount San Antonio, 22 miles away. The distance was measured by the United States Coast and Geodetic Survey with an accuracy of two parts in a million.

The source of the ray was a powerful electric arc lamp giving a light almost as bright as the sun. Passing through a minute hole in front of the lamp the ray was caught on a revolving octagonal mirror, sent to Mount San Antonio, reflected back from there and received on the original mirror which was revolved at such a rate as to catch the returned ray on the succeeding face of the octagon. The mirror, rotated by a blast of air playing on a little windmill, made 530 revolutions a second, its speed being regulated by a tuning fork of known pitch.

The average results of eight observations, as reported by Professor Michelson at the recent Franklin Institute Centenary, give the velocity of light in a vacuum as 186,300 miles per second, which is probably accurate to within 20 miles. Next summer he hopes that it will be possible, by extending the distance to 100 miles, to get the figure accurate to within one part of a million. (News letter of University of Chicago.)

IN referring to a student who stood high in his algebra class, and who was not able to do a simple problem in that subject, A. W. Forbes says:[1]

"He did receive practice in memorizing meaningless symbols, the same kind of practice as he would have if he had learned Chinese words without bothering to learn their meaning. Is this kind of practice desirable? I am inclined to think that learning Chinese words would have been better education, for

[1] "Mathematics or Hieroglyphics?" *Education*, October, 1924.

it would have been frankly only an exercise, while in the case of algebra he was getting a false view of mathematics. He was learning to think of mathematics as a mass of symbols juggled by cranks, instead of a method of expressing his thoughts more easily and clearly than can be done with ordinary words. . . . It is better to teach a little mathematics than a lot of hieroglyphics."—Alfred Davis.

At the meeting of the mathematics and science section of the Washington Educational Association held at Walla Walla October 27-29, four papers were presented, two of which were on mathematical topics. Miss Theresa Tromp, of the Walla Walla High School, gave a very illuminating talk on "The Fourth Dimension and Hyperspace." Miss Edith Greenberg, of North Central High School, Spokane, spoke very enthusiastically of the successful experiment with general mathematics in her own high school.—W. C. Eells.

The Chicago Mathematics Club has been in existence for eleven years. The club was started by C. M. Austin of Oak Park in January, 1914. The Presidents with their term of office are given herewith: C. M. Austin, Oak Park, 1914-1916; John R. Clark, Chicago Normal, 1916-1918; W. W. Gorsline, Crane Junior College, 1918-1920; M. J. Newell, Evanston, 1920-1921; H. C. Wright, University High, 1921-1922; Everett Owen, Oak Park, 1922-1923; Olice Winter, Harrison Tech., 1923-1924; E. L. Schrieber, Proviso Township, 1924-1925.

The programs for 1923-1924 were as follows:

October 26th—Election of officers: President, Olice Winter; Secretary-Treasurer, Marx Holt; Recording Secretary, Edwin W. Schreiber. "Mathematics Classroom Technique." Prof. C. W. Meyer, Marx Holt, A. M. Allison, C. E. Kitch. Attendance 39.

November 16th—"Some Mathematical Aspects of Modern Astronomy," Professor F. R. Moulton, University of Chicago. Attendance 41.

December 14th—"Resolved, That the Ohio Requirements for Mathematics in High Schools Be Adopted in the Chicago Area."

Affirmative, B. Laughlin and J. C. Piety; negative, S. J. A. Conner and O. M. Miller. Attendance 34.

January 18th—"Mathematical Phases of Modern Telephone Work," Mr. Wilber Roadhouse, Transmission Engineer, Am. Tel. and Tel. Co. Attendance 34.

February 23rd—Joint meeting with National Council of Teachers of Mathematics.

March 21st—"Mathematical Instruments of 200 Years Ago." Illustrated with lantern slides. Edwin W. Schreiber, Proviso. Attendance 31.

April 17th—"Grand Pow Wow on the Report of the National Committee." Representatives from Crane, Evanston, and Oak Park. Attendance 30.

May 16th—"Algebra." Supt. E. E. Arnold, The Pelhams, New York. A joint meeting with the Women's Mathematics Club of Chicago. Attendance 28. (Courtesy E. L. S.)

SO LET ME TEACH

So let me teach from day to day
That those who 'round me toil and play
Shall as the shadows come and go
Remember me, not what they know.

So let me teach from week to week
That those who come and wisdom seek
Shall when their laurels they have earned
Remember me, not lessons learned.

So let me teach from time to time
That those who gather at my shrine
Shall when the bloom has left their cheeks
Remember me, not words or books.

So let me teach throughout the years
That those Ive seen in smiles and tears
Shall when they seek my door no more
Remember me, not four time four.

W. L. H.

ANNUAL MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The 1925 meeting of the National Council of Mathematics Teachers will be held at Cincinnati Saturday, February 21st. The Department of Superintendence meets in Cincinnati the following week. All of the mathematics meetings will be held in the Ball Room of the Sinton Hotel. There will be a dinner for all members at the Sinton Hotel at six-thirty. Also there will be a luncheon at twelve o'clock noon for members of the Executive Committee.

As may be observed, the emphasis of each program is upon investigational and experimental phases. The Cincinnati meeting should prove to be a step forward in the teaching of mathematics. There may be minor changes, but the following is substantially correct:

MORNING PROGRAM

Ballroom, Hotel Sinton, Ten O'Clock

(a) The Measured Results of the Wisconsin Supervised Study Program in Mathematics
Professor Walter W. Hart, University of Wisconsin

(b) The Possibility of Conceptualizing the Processes of Thinking as They Occur in Plane Geometry
Miss Winona Perry, The Lincoln School

(c) Individual Instruction in Ninth Grade Algebra
Mr. C. M. Stokes, New Trier Township High School, Kenilworth, Illinois

(d) Annual Statement by the Editor of the *Mathematics Teacher*
Dr. John R. Clark, The Lincoln School of Teachers College

(e) What Algebra Is Retained by College Freshmen?
Professor Walter Crosby Eells, Whitman College, Walla Walla, Washington

(f) Brief Business Meeting and Appointment of Nominating Committee

AFTERNOON PROGRAM

Ballroom, Sinton Hotel, Two O'Clock

(a) The Psychological Approach to Curriculum Construction in High School Mathematics
Dr. J. Worth Osburn, Director of Educational Measurements, Dept. of Public Instruction, State of Wisconsin

(b) The Problem of Drill in the Seventh and Eighth Grades
Professor O. S. Lutes, State University of Iowa

(c) The Relation of Standard Tests to the Re-Organization of Mathematics
Professor Clifford Brewster Upton, Teachers College, Columbia University

EVENING PROGRAM

Dinner For All Members, Ballroom, Sinton Hotel, Six-Thirty

(a) The Content and Method of the New Curriculum in Arithmetic
Dr. G. M. Wilson, Boston University

(b) Factors of Success in Ninth Grade Algebra
Edwin L. Schreiber, Head of Mathematics Department,
Proviso Township High School, Maywood, Ill.

(c) The Psychological Analysis of a Section of Algebra
Professor F. B. Knight, The State University of Iowa

It is probable that Professor Herbert E. Slaught of the University of Chicago will attend and serve as toastmaster at the dinner program.

RALEIGH SCHORLING,
President, National Council of Mathematics Teachers.

NEW BOOKS

Teaching Junior High School Mathematics. H. C. BARBER. Houghton-Mifflin Co., Boston, 1924; pp. 136.

The curriculum in mathematics in grades seven, eight and nine has, during the last ten years, been extensively modified. The best evidence of this fact is the eight or ten series of text books in junior high school mathematics that have been published during this period. In these texts there has been a revaluation of the aims of mathematical instruction. The first book on the teaching of junior high school mathematics discusses with unusual clarity and sanity the new attitude toward arithmetic, algebra and geometry in these grades. This little monograph, the first of a series of mathematical monographs being prepared under the editorial direction of Prof. John W. Young, tells what the "new program" in mathematics is, why it should replace the "older program," and makes numerous suggestions on methods of teaching the newer materials. The discussions of the place of algebra and geometry in the seventh and eighth grades, the "new algebra" in the ninth grade, and approximate computation, interpret the spirit and purpose of the junior high school mathematics most admirably. The author considers the purposes of mathematics in these grades to be the development of (1) a rational, problem-solving attitude, (2) skill in applying arithmetic to the every day problem of the home, store and farm, (3) the ability to compute with "approximate data," (4) an appreciation of, and some skill in using, the algebraic formula and the equation, and (5) a knowledge of certain properties of space (intuitive geometry) that is to be acquired by the experimental laboratory method.

The section in which a hypothetical pupil summarizes his conception of the meaning and uses of seventh and eighth grade mathematics is an excellent concerete statement of the ideals of the new program in mathematics. If courses in mathematics are to be elective beyond the junior high school years, is it desirable to include in the ninth grade an introduction to demonstrative geometry? Mr. Barber is not certain, but he thinks *yes*. If time is not available for both numerical trigonometry

and an introduction to demonstrative geometry, he prefers the trigonometry.

The monograph is particularly deserving of commendation for the success with which it interprets the meaning and purpose of the "new mathematics." The section on computation will be read and re-read by teachers who desire to take a more common sense attitude toward computation in general, and who wish to familiarize themselves with the technique of approximate computation. It is in no sense a disparagement of the book to note that the author has given little consideration to the methods of fixing skills or problem solving. The book will be very influential in helping both teachers and laymen understand and approve the type of mathematics which it describes.